

# CYCLICITY OF THE LEFT REGULAR REPRESENTATION OF A LOCALLY COMPACT GROUP

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**ABSTRACT.** We combine harmonic analysis and operator algebraic techniques to give a concise argument that the left regular representation of a locally compact group is cyclic if and only if the group is first countable, a result first proved by Greenleaf and Moskowitz.

Let  $G$  be a locally compact group and let  $\lambda$  and  $\rho$  denote the (unitarily equivalent) *left and right regular representations* of  $G$  on  $L^2(G)$ , respectively. The *group von Neumann algebra*  $VN(G)$  is the von Neumann algebra generated in  $B(L^2(G))$  by  $\lambda(G)$ . It is well known that the commutant of  $VN(G)$  is the von Neumann algebra generated by  $\rho(G)$ . In [2], an operator algebraic argument viewing  $VN(G)$  as arising from a left Hilbert algebra, in combination with a reduction argument using the structure theory of locally compact groups, is used to show that  $\lambda$  is cyclic when the group  $G$  is first countable. The converse, left open in [2], was established later by the same authors in [3]. The purpose of this note is to give a new and more economical proof of this equivalence. Moreover, we show that these conditions are equivalent to  $\sigma$ -finiteness of  $VN(G)$ , the latter condition arising naturally from our techniques. In the commutative case, it is well known that  $\sigma$ -finiteness of  $L^\infty(G)$  characterizes  $\sigma$ -compactness of  $G$ , and it is our hope that further development of the techniques we employ will yield natural characterizations of  $\sigma$ -finiteness of a general locally compact quantum group. An alternative proof of the characterization we establish, exploiting the structure theory of locally compact groups, is given in [5].

Recall that the *support* of a normal state  $\omega$  on a von Neumann algebra  $M$  is the minimal projection  $S_\omega$  in  $M$  for which  $\langle \omega, S_\omega \rangle = 1$  and that  $\omega$  is *faithful* if  $S_\omega = I$ , the identity in  $M$ , equivalently if  $\omega$  takes strictly positive values on strictly positive operators. We record some elementary facts about these concepts. For a vector  $\xi$  in a Hilbert space, let  $\omega_\xi$  denote the vector functional implemented by  $\xi$ . The notation  $\langle X \rangle$  denotes the norm closed linear span of  $X$ .

**Lemma 0.1.** *Let  $\mathcal{H}$  be a Hilbert space, let  $M$  a von Neumann algebra in  $B(\mathcal{H})$ , and let  $\xi, \eta \in \mathcal{H}$  be unit vectors. The following hold.*

- (1) *The projection  $S_{\omega_\xi}$  has range  $\langle M'\xi \rangle$ .*
- (2)  *$\langle \omega_\xi, S_{\omega_\eta} \rangle = 0$  if and only if  $\xi$  is orthogonal to  $\langle M'\eta \rangle$ .*
- (3) *A projection  $P$  in  $M$  satisfies  $P\xi = \xi$  if and only if  $S_{\omega_\xi} \leq P$ .*
- (4) *A normal state  $\omega$  on  $M$  is faithful if and only if  $\langle \omega, U \rangle = \langle \omega, I \rangle$  implies  $U = I$ , for any unitary  $U$  in  $M$ .*

Motivated by the following simple observation, we choose to characterize cyclicity of the right regular representation.

**Lemma 0.2.** *Let  $G$  be a locally compact group. A vector  $\xi \in L^2(G)$  is cyclic for  $\rho$  if and only if  $\omega_\xi$  is faithful on  $VN(G)$ .*

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*Proof.* For  $\xi \in L^2(G)$  we have  $\langle \rho(G)\xi \rangle = \langle VN(G)'\xi \rangle$  since  $\text{span}\rho(G)$  is strong operator topology dense in  $VN(G)'$ . Consequently, the vector  $\xi$  is cyclic for  $\rho$  if and only if  $\langle VN(G)'\xi \rangle = L^2(G)$ . As  $S_{\omega_\xi}$  has range  $\langle VN(G)'\xi \rangle$ , the latter assertion is exactly that  $S_{\omega_\xi} = I$ .  $\square$

Let  $A(G)$  denote the *Fourier algebra* of a locally compact group  $G$ , which is the predual of  $VN(G)$ , and for  $T \in VN(G)$  and  $u \in A(G)$  define  $T \cdot u \in A(G)$  by

$$\langle T \cdot u, S \rangle = \langle u, \check{T}S \rangle \quad (S \in VN(G)),$$

where  $\check{T}$  is the image of  $T$  under the adjoint of the check map  $u \mapsto \check{u}$  on  $A(G)$  (here,  $\check{u}(s) = u(s^{-1})$ ). See [1, p.213]. Proposition 3.17 of [1] shows that for  $u \in A(G) \cap L^2(G)$  we have  $T \cdot u = Tu$ , where the right hand side is evaluation of the operator  $T$  at the vector  $u$  in  $L^2(G)$ . This fact is needed in the following lemma, which is key to establishing the main result.

**Lemma 0.3.** *Let  $G$  be a locally compact group. Every nonzero projection in  $VN(G)$  has a nonzero continuous function in its range.*

*Proof.* Let  $P \in VN(G)$  be a nonzero projection and choose a unit vector  $\xi$  in its range. Since positive functions span  $\mathcal{C}_c(G)$ , which is in turn dense in  $L^2(G)$ , we may find a positive  $f \in \mathcal{C}_c(G)$  of norm one in  $L^2(G)$  and not orthogonal to  $\langle \rho(G)\xi \rangle$ , so that  $\langle \omega_f, S_{\omega_\xi} \rangle \neq 0$  by Lemma 0.1. The function  $\omega_f$  in  $A(G)$  is positive definite and pointwise positive, so that  $\check{\omega}_f = \omega_f$ , and is in  $A(G) \cap L^2(G)$  because  $f$  has compact support, whence

$$S_{\omega_\xi}(\omega_f)(e) = (S_{\omega_\xi} \check{\omega}_f)(e) = \langle S_{\omega_\xi} \check{\omega}_f, \lambda(e) \rangle = \langle \omega_f, S_{\omega_\xi} \rangle = \langle \check{\omega}_f, S_{\omega_\xi} \rangle = \langle \omega_f, S_{\omega_\xi} \rangle \neq 0.$$

Thus  $S_{\omega_\xi}(\omega_f) = S_{\omega_\eta} \check{\omega}_f$  is nonzero and in  $A(G)$ , hence continuous, and is in the range of  $P$  because  $S_{\omega_\xi} \leq P$ , by Lemma 0.1.  $\square$

**Theorem 0.4.** *Let  $G$  be a locally compact group. The following are equivalent:*

- (1)  $G$  is first countable.
- (2)  $VN(G)$  is  $\sigma$ -finite.
- (3) The left (equivalently, right) regular representation is cyclic.

*Proof.* Suppose (1) holds. Let  $(U_n)_{n=1}^\infty$  be a countable neighborhood base at the identity in  $G$  and define  $\omega_n = |U_n|^{-1} \omega_{\chi_{U_n}}$ . We show that the normal state  $\omega = \sum_{n=1}^\infty 2^{-n} \omega_n$  is faithful. Let  $T$  be a positive operator in  $VN(G)$  with  $\langle \omega, T \rangle = 0$  and let  $P$  be the range projection of  $T$ , so that  $\langle \omega, P \rangle = 0$  (see, e.g., [4, Remark 7.2.5]). Given any vector  $\eta$  in the range of  $T$  we have  $S_{\omega_\eta} \leq P$  and thus  $0 \leq \langle \omega_n, S_{\omega_\eta} \rangle \leq \langle \omega_n, P \rangle \leq \langle \omega, P \rangle = 0$ , implying that  $\eta$  is orthogonal to  $\langle \rho(G)\chi_{U_n} \rangle$  for each  $n \geq 1$ . If  $\eta$  is continuous, then

$$\eta(s) = \lim_n |U_n s|^{-1} \int_{U_n s} \eta = \lim_n |U_n s|^{-1} \langle \eta | \rho(s^{-1}) \chi_{U_n} \rangle \Delta(s)^{\frac{1}{2}} = 0$$

for every  $s \in G$ . Thus  $P = 0$  by Lemma 0.3, hence  $T = 0$  and  $\omega$  is faithful, so (2) holds.

Normal states on  $VN(G)$ , being positive definite functions in  $A(G)$ , are vector states, so that statements (2) and (3) are equivalent by Lemma 0.2.

We provide the argument of [6] establishing that (2) implies (1). Suppose (2) holds and let  $\omega$  be a faithful normal state on  $VN(G)$ . Fix a compact neighborhood  $K$  of the identity in  $G$  and let  $V$  be any open neighborhood of the identity contained in  $K$ . We show that the sets  $U_n = \{s \in K : |\omega(s) - 1| < \frac{1}{n}\}$  form a neighborhood base at the identity, for which it suffices to establish that  $U_n$  is contained in  $V$  for some  $n \geq 1$ . For any  $s \in G$  with  $\omega(s) = 1$ , Lemma 0.1 entails that  $s = e$ , since  $\omega(s) = \langle \omega, \lambda(s) \rangle$ . Compactness of  $K \setminus V$  then implies that  $\epsilon = \inf \{|\omega(s) - 1| : s \in K \setminus V\}$  is strictly positive. Choosing  $N \geq 1$  with  $\frac{1}{N} < \epsilon$ , if  $s \in U_N$ , then that  $s \in K$  and  $|\omega(s) - 1| < \epsilon$  together imply that  $s \in V$ . Thus  $U_N \subset V$ , as required.  $\square$

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